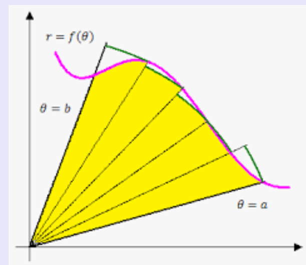
Calculus II

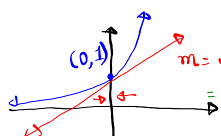
Lecture 2



Feb 19-8:47 AM

Class QZ 1

Find the equation of the tangent line to the graph $\boxed{f(x)=e^x}$ at $x=0$. Final answer in Slope-Int. Form



$m = f'(0) = e^0 = 1$

$f(x)=e^x$

$f'(x)=e^x$

$y - y_1 = m(x - x_1)$

$y - 1 = 1(x - 0) \rightarrow \boxed{y = x + 1}$

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = 0$

$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = 1$

$\Rightarrow \lim_{x \to 0} f(x) = 1$

Since $\lim_{x \to 0} f(x) = f(0)$, then $f(x)$ is Cont. at $x=0$.

$f(x)=e^x$ is Cont. everywhere

$f'(x)=e^x > 0$, $f(x)$ is increasing. $\Rightarrow$ $f(x)$ is one-to-one

$f(x)$ has an inverse.

$e = \lim_{h \to 0} \frac{e^h - 1}{h}$, $\frac{d}{dx}[e^x] = e^x$, $\int e^x dx = e^x + C$

Find $\frac{dy}{dx}$ for $y = \cot(e^{\boxed{2x}} - 2)$.

$\frac{dy}{dx} = -\csc^2(e^{2x} - 2) \cdot [e^{2x} \cdot 2 - 0]$

$$\boxed{\frac{dy}{dx} = -2e^{2x} \csc^2(e^{2x} - 2)}$$

Evaluate $\int e^{3x} \csc^2(e^{3x}) dx$

Hint: Let $u = e^{3x}$

$du = e^{3x} \cdot 3\, dx$

$\frac{du}{3} = e^{3x} dx$

$= \int \csc^2 u \frac{du}{3} = \frac{1}{3} \int \csc^2 u\, du$

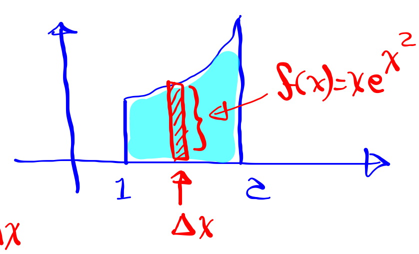
$= -\frac{1}{3} \cot u + C$

$$= \boxed{-\frac{1}{3} \cot e^{3x} + C}$$

Find the area below $f(x) = x e^{x^2}$, above the x-axis, from $x = 1$ to $x = 2$.

Draw, Shade, give Ref. Rect.

$f(x) \geq 0$ on $[1, 2]$



$A = \int_1^2 x e^{x^2} dx$

$u = x^2$

$du = 2x\, dx$

$\frac{du}{2} = x\, dx$

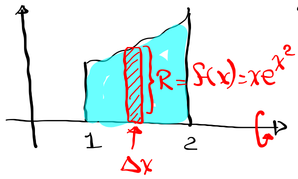
$x = 1 \quad u = 1$

$x = 2 \quad u = 4$

$= \int_1^4 e^u \frac{du}{2}$

$$= \frac{1}{2} e^u \Big|_1^4 = \frac{1}{2}[e^4 - e^1] = \boxed{\frac{e^4 - e}{2}}$$

Rotate the region from last example about x-axis, find its volume.



Ref. Rect. $\perp$ to A.O.R.
Region is 100% attached to A.O.R.
Disk Method

$$V=\int_1^2 \pi [R]^2 dx = \pi \int_1^2 \left[x e^{x^2}\right]^2 dx = \pi \int_1^2 x^2 e^{2x^2} dx$$

Suppose $f(x)=\sqrt{x}\, e^{x^2}$

I need to know integration by parts.

$$V=\int_1^2 \pi \left(\sqrt{x}\, e^{x^2}\right)^2 dx$$

$$=\pi \int_1^2 x\, e^{2x^2} dx$$

Recall $(x^m)^n = x^{mn}$

$u=2x^2$

$du=4x\,dx$

$\frac{du}{4}=x\,dx$

$$=\pi \int_2^8 e^u \frac{du}{4}$$

$$=\frac{\pi}{4} e^u \Big|_2^8 = \frac{\pi(e^8-e^2)}{4}$$

Given $f(x)=3x+\sin(x-1)$

1) Show that $f(x)$ is one-to-one function.

Hint: use $f'(x)$

If $f'(x)\ge 0 \to f(x)$ increasing $\to f(x)$ is one-to-one

$f'(x)=3+\cos(x-1)\cdot 1$

$-1\le \cos\alpha \le 1$

$-1\le \cos(x-1)\le 1$

$3-1\le 3+\cos(x-1)\le 3+1$

$\to 2\le f'(x)\le 4$

When $f(x)$ is one-to-one, then $f^{-1}(x)$ exists.

Evaluate $(f^{-1}(x))'(3)$

$f^{-1}(3)=1$ $\quad f(1)=3$

$f(\,?\,)=3$

$f(x)=3x+\sin(x-1)$

$f(1)=3(1)+\sin(1-1)$

$=3+\cancel{\sin 0}^{\,0}=3$

Recall

$$(f^{-1})'(3)=\frac{1}{f'(f^{-1}(3))}$$

$$=\frac{1}{f'(1)}=\boxed{\frac{1}{4}}$$

$f(x)=3x+\sin(x-1)$

$f'(x)=3+\cos(x-1)$

$f'(1)=3+\cancel{\cos 0}^{\,1}=4$

Find $\frac{dy}{dx}$ for $y = \sin(e^x) + e^{\sin x}$

$$\boxed{y' = \cos(e^x)\cdot e^x + e^{\sin x}\cdot \cos x}$$

Evaluate $\int (e^x + e^{-x})^2\, dx$

$x^{-n} = \frac{1}{x^n}$

$(A+B)^2 = A^2 + 2AB + B^2$

$(x^m)^n = x^{mn}$

$= \int \left[(e^x)^2 + 2(e^x)(e^{-x}) + (e^{-x})^2 \right] dx$ $\quad e^0 = 1$

$= \int \left[e^{2x} + 2 + e^{-2x} \right] dx$

$= \int e^{2x} dx + \int 2\, dx + \int e^{-2x} dx$

$= \frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} + C$

How do we verify this?

$\frac{d}{dx}\left[\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + C\right] = \frac{1}{2}\cdot e^{2x}\cdot 2 + 2 - \frac{1}{2}e^{-2x}\cdot(-2)$

$= e^{2x} + 2 + e^{-2x} = (e^x + e^{-x})^2$

Logarithmic Functions:

$$\log_a x = y \iff a^y = x$$

$a > 0,\ a \neq 1$

$x > 0$

$$\log_3 x = y \iff 3^y = x$$

Natural log $\rightarrow$ base $= e$ $\rightarrow \log_e x = \ln x$

Common log $\rightarrow$ base $= 10$ $\rightarrow \log_{10} x = \log x$

Suppose $f(x) = \ln x$

Find $f'(x)$.

Let $y = f(x)$

$y = \ln x$

$\ln x = y \Rightarrow \log_e x = y$

$\Rightarrow e^y = x$

$$\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$$

$$e^y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\int \frac{1}{x}\,dx = \ln x + C$$

$x > 0$

$$\boxed{\begin{array}{l} \frac{d}{dx}[\ln x] = \frac{1}{x} \\ \int \frac{1}{x}\,dx = \ln x + C, \ x>0 \end{array}}$$

find $\frac{dy}{dx}$

$$\frac{d}{dx}[\sqrt{x}] = \frac{d}{dx}[x^{1/2}]$$

1) $y = \underbrace{x \ln x}_{\text{Product Rule}} + e^{\sqrt{x}}$

$$y' = 1 \cdot \ln x + x \cdot \frac{1}{x} + e^{\sqrt{x}} \cdot \frac{1}{2} x^{\frac{1}{2}-1}$$

$$\boxed{y' = \ln x + 1 + \frac{e^{\sqrt{x}}}{2\sqrt{x}}}$$

2) $y = \dfrac{1}{\ln x^2}$

Recall

$\log_b x^m = m \log_b x$

$\dfrac{1}{x} = x^{-1}$

$y = \dfrac{1}{2\ln x}$

$y = \frac{1}{2}(\ln x)^{-1}$

$\dfrac{dy}{dx} = \frac{1}{2} \cdot -1 \cdot (\ln x)^{-2} \cdot \frac{1}{x}$

$\dfrac{dy}{dx} = \dfrac{-1}{2x(\ln x)^2}$

find $f'(1)$ if $f(x) = \dfrac{x}{1 + \ln x}$

$$f'(x) = \frac{\frac{d}{dx}[x] \cdot (1 + \ln x) - x \cdot \frac{d}{dx}[1 + \ln x]}{[1 + \ln x]^2}$$

$$= \frac{1(1 + \ln x) - x \cdot \frac{1}{x}}{[1 + \ln x]^2} = \frac{\ln x}{[1 + \ln x]^2}$$

$$f'(1) = \frac{\cancelto{0}{\ln 1}}{[1 + \cancelto{0}{\ln 1}]^2} = \frac{0}{1} = \boxed{0}$$

Evaluate $\int_1^4 \frac{2}{x}\,dx = 2\int_1^4 \frac{1}{x}\,dx$

$= 2\cdot \ln x \Big|_1^4 = 2(\ln 4 - \cancel{\ln 1}^{\,0})$

$= \boxed{2\ln 4}$

Find $f'(x)$ for $f(x) = \ln\left[x + \sqrt{x^2-1}\right]$

Recall

$\frac{d}{dx}[\ln u] = \frac{1}{u}\cdot\frac{du}{dx}$

$\boxed{f'(x) = \frac{1}{x+\sqrt{x^2-1}}\cdot\left[1 + \frac{x}{\sqrt{x^2-1}}\right]}$

$f'(x) = \frac{1}{\cancel{x+\sqrt{x^2-1}}}\left[\frac{\cancel{\sqrt{x^2-1} + x}}{\sqrt{x^2-1}}\right]$

$\frac{d}{dx}\left[\sqrt{x^2-1}\right] =$

$\frac{d}{dx}\left[(x^2-1)^{1/2}\right] =$

$\frac{1}{2}(x^2-1)^{-1/2}\cdot 2x =$

$\frac{x}{\sqrt{x^2-1}}$

$\boxed{f'(x) = \frac{1}{\sqrt{x^2-1}}}$

Evaluate $\int_1^4 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

Use $(A+B)^2 = \text{- - - -}$

$A^2 + 2AB + B^2$

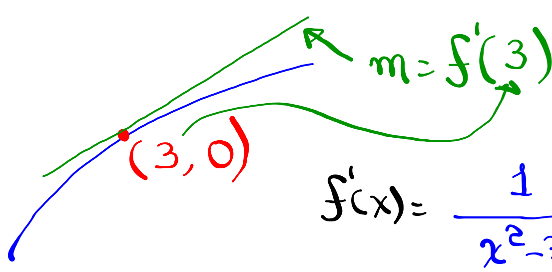
$$= \int_1^4 \left[x + 2\cancel{\sqrt{x}} \cdot \frac{1}{\cancel{\sqrt{x}}} + \frac{1}{x}\right] dx$$

$$= \left(\frac{x^2}{2} + 2x + \ln x\right)\Big|_1^4$$

$$= (8 + 8 + \ln 4) - \left(\frac{1}{2} + 2 + \cancelto{0}{\ln 1}\right)$$

$$= 14 + \ln 4 - \frac{1}{2} = \boxed{13.5 + \ln 4}$$

Find equation of tan. line to the graph of $f(x) = \ln(x^2 - 3x + 1)$ at $x = 3.$



$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx}$

$$f'(x) = \frac{1}{x^2 - 3x + 1} \cdot (2x - 3)$$

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 3(x - 3)$$

$$\boxed{y = 3x - 9}$$

$$f'(3) = \frac{2(3) - 3}{3^2 - 3(3) + 1} = \frac{3}{1} = \boxed{3}$$

find the volume if the region bounded

$f(x) = \frac{1}{\sqrt{x+1}}$, x-axis, y-axis, and $x = 3$

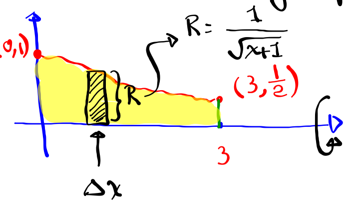
is rotated about x-axis. Drawing Required.

$f(x) > 0$

y-axis $\to x = 0$

$f(0) = 1$

$f(3) = \frac{1}{2}$



Disk

$$V = \int_0^3 \pi \left[\frac{1}{\sqrt{x+1}}\right]^2 dx = \pi \int_0^3 \frac{1}{x+1}\, dx$$

$u = x + 1$

$du = dx$

$$= \pi \int_1^4 \frac{1}{u}\, du = \pi \ln u \Big|_1^4$$

$$= \boxed{}$$

find $\frac{dy}{dx}$ if $y = \ln\left(x^2 + y^2\right)$

$$\frac{dy}{dx} = \frac{1}{x^2 + y^2} \cdot \left(2x + 2y \cdot \frac{dy}{dx}\right)$$

$$(x^2 + y^2)\frac{dy}{dx} = 2x + 2y\frac{dy}{dx}$$

$$(x^2 + y^2)\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x$$

$\vdots$

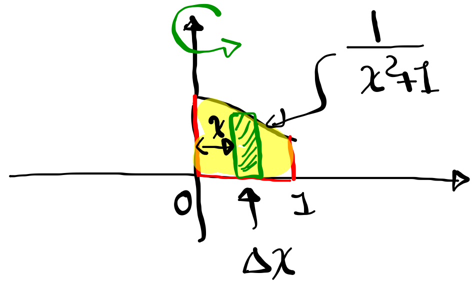
$$\frac{dy}{dx} = \frac{2x}{x^2 + y^2 - 2y}$$

Consider the region bounded by

$y = \frac{1}{x^2+1}$, $x = 0$, $y = 0$, and $x = 1$.

Rotate about Y-axis,

find the Volume.



Shell

$$V = \int_0^1 2\pi(x)\left(\frac{1}{x^2+1}\right)dx$$

$u = x^2+1$

$du = 2x\,dx$

$$= \pi\int_1^2 \frac{1}{u}\,du = \pi \ln u\Big|_1^2 = \boxed{\quad}$$

Find $\frac{dy}{dx}$ if $y = \ln(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x}\cdot(\sec x\tan x + \sec^2 x)$$

$\frac{d}{dx}[\ln u] = \frac{1}{u}\cdot\frac{du}{dx}$

$$= \frac{\sec x\,\cancel{(\tan x + \sec x)}}{\cancel{\sec x + \tan x}} = \boxed{\sec x}$$

$$\int \sec x\,dx = \ln(\sec x + \tan x) + C$$

Evaluate $\int_1^e \frac{\sin(\ln x)}{x}\,dx$

Hint:

Let $u = \ln x$

$du = \frac{1}{x}\,dx$

$= \int_0^1 \sin u \, du = -\cos u \Big|_0^1$

$= -\left[\cos 1 - \cancel{\cos} 0\right]$ 1

$= -\cos 1 + 1 = \boxed{1 - \cos 1}$

Radian

Looking Ahead

Differential Equation

Find y if $\frac{dy}{dx} = xe^{-y}$

$\frac{dy}{dx} = \frac{x}{e^y}$, $e^y dy = x\,dx$

$\int e^y dy = \int x\,dx$

$e^y = \frac{x^2}{2} + C$

Isolate y → $\ln e^{\boxed{y}} = \ln\left[\frac{x^2}{2} + C\right]$

$y \cancel{\ln e} = \ln\left(\frac{x^2}{2} + C\right)$ 1

$y = \ln\left[\frac{1}{2}x^2 + C\right]$